

Function of two variables: — Let u be a function of two independent variables x and y which assume arbitrarily chosen values independent of each other.

Let x assume any value lying in a given interval $[a, b]$ and for each value of x , the variable y may have any value lying in any given interval $[c, d]$. Thus a system of ordered pairs (x, y) is obtained and to each of the possible pairs (x, y) , there corresponds a unique value of u . The aggregate of the pairs of numbers (x, y) for which the function is defined is called the domain of the function. Here the domain is rectangular and is symbolically represented by $R(a, b; c, d)$.

Double or Simultaneous Limit: —

Definition: — A function $u = f(x, y)$ is said to have a double limit l as the point $(x, y) \rightarrow (a, b)$, if for any arbitrary small positive number ϵ , there exists a positive number δ , such that

$$|f(x, y) - l| < \epsilon \text{ whenever } |x - a| \leq \delta \text{ and } |y - b| \leq \delta.$$

The double limit is written as

$$\lim_{(x, y) \rightarrow (a, b)} f(x, y) = l.$$

$$(x, y) \rightarrow (a, b)$$

Here $(x, y) \rightarrow (a, b)$ means that $x \rightarrow a, y \rightarrow b$.

The double limit can also be written in the alternative form as

$$\lim_{\substack{x \rightarrow a \\ y \rightarrow b}} f(x, y) = l.$$

Examples — Evaluate $\lim_{(x, y) \rightarrow (0, 0)} \left[\frac{xy}{\sqrt{x^2 + y^2}} \right]$.

Solution: — Let $y = mx$.

$$\text{Then } f(x, y) = \frac{x \cdot mx}{\sqrt{x^2 + m^2 x^2}} = \frac{mx}{\sqrt{1 + m^2}}$$

$$\therefore \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} f(x, y) = \lim_{x \rightarrow 0} \frac{mx}{\sqrt{1 + m^2}} = 0$$

for all finite values of m .

Repeated limits: — let the function $u = f(x, y)$ be defined in a certain nbd. of the point (a, b)

Then $\lim_{y \rightarrow b} f(x, y)$ is a function of x ,

say $\phi(x)$ provided it exists.

if now $\lim_{x \rightarrow a} \phi(x)$ exists finitely or ∞ , infinitely

say, $\lim_{x \rightarrow a} \phi(x) = l$, then

We write

$$\lim_{x \rightarrow a} \lim_{y \rightarrow b} f(x, y) = l$$

Which is called the repeated limit of $f(x, y)$ at the point (a, b) ; the limit for $y \rightarrow b$ is taken first and then afterwards for $x \rightarrow a$.